

Nabla Fractional Calculus on Time Scales and Inequalities

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Abstract

Here we develop the Nabla Fractional Calculus on Time Scales. Then we produce related integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert-Pachpatte. Finally we give inequalities applications on the time scales \mathbb{R} , \mathbb{Z} .

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1 Background and Foundation Results

For the basics on time scales we follow [1], [2], [3], [4], [9], [11], [13], [6], [7], [10].

By [15], p. 256, for $\mu, \nu > 0$ we have that

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \quad (1)$$

where Γ is the gamma function.

Here we consider time scales T such that $T_k = T$.

Consider the coordinatewise ld-continuous functions $\widehat{h}_\alpha : T \times T \rightarrow \mathbb{R}$, $\alpha \geq 0$, such that $\widehat{h}_0(t, s) = 1$,

$$\widehat{h}_{\alpha+1}(t, s) = \int_s^t \widehat{h}_\alpha(\tau, s) \nabla \tau, \quad (2)$$

$\forall s, t \in T$.

Here ρ is the backward jump operator and $\nu(t) = t - \rho(t)$.

Furthermore for $\alpha, \beta > 1$ we assume that

$$\int_{\rho(u)}^t \widehat{h}_{\alpha-1}(t, \rho(\tau)) \widehat{h}_{\beta-1}(\tau, \rho(u)) \nabla \tau = \widehat{h}_{\alpha+\beta-1}(t, \rho(u)), \quad (3)$$

valid for all $u, t \in T : u \leq t$.

In the case of $T = \mathbb{R}$; then $\rho(t) = t$, and $\widehat{h}_k(t, s) = \frac{(t-s)^k}{k!}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and define

$$\widehat{h}_\alpha(t, s) = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

Notice that

$$\int_s^t \frac{(\tau-s)^\alpha}{\Gamma(\alpha+1)} d\tau = \frac{(t-s)^{\alpha+1}}{\Gamma(\alpha+2)} = \widehat{h}_{\alpha+1}(t, s),$$

fulfilling (2).

Furthermore we observe that $(\alpha, \beta > 1)$

$$\begin{aligned} \int_u^t \widehat{h}_{\alpha-1}(t, \tau) \widehat{h}_{\beta-1}(\tau, u) d\tau &= \int_u^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(\tau-u)^{\beta-1}}{\Gamma(\beta)} d\tau \\ &\stackrel{(by (1))}{=} \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = \widehat{h}_{\alpha+\beta-1}(t, u), \end{aligned}$$

fulfilling (3).

By Theorem 2.2 of [14], we have for $k, m \in \mathbb{N}_0$ that

$$\int_{t_0}^t \widehat{h}_k(t, \rho(\tau)) \widehat{h}_m(\tau, t_0) \nabla \tau = \widehat{h}_{k+m+1}(t, t_0). \quad (4)$$

Let $T = \mathbb{Z}$, then $\rho(t) = t - 1$, $t \in \mathbb{Z}$. Define $t^{\bar{0}} := 1$, $t^{\bar{k}} := t(t+1)\dots(t+k-1)$, $k \in \mathbb{N}$, and by (2) we have $\widehat{h}_k(t, s) = \frac{(t-s)^{\bar{k}}}{k!}$, $s, t \in \mathbb{Z}$, $k \in \mathbb{N}_0$.

Here $\int_{t_0}^t \nabla \tau = \sum_{\tau=t_0+1}^t$.

Therefore by (4) we get

$$\sum_{\tau=t_0+1}^t \frac{(t-\tau+1)^{\bar{k}}}{k!} \frac{(\tau-t_0)^{\bar{m}}}{m!} = \frac{(t-t_0)^{\bar{k+m+1}}}{(k+m+1)!},$$

which results into

$$\sum_{\tau=t_0}^t \frac{(t-\tau+1)^{\bar{k-1}}}{(k-1)!} \frac{(\tau-t_0+1)^{\bar{m-1}}}{(m-1)!} = \frac{(t-t_0+1)^{\bar{k+m-1}}}{(k+m-1)!}, \quad (5)$$

confirming (3).

Next we follow [5].

Let $a, \alpha \in \mathbb{R}$, define $t^{\bar{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$, $t \in \mathbb{R} - \{-2, -1, 0\}$, $N_a = \{a, a \pm 1, a \pm 2, \dots\}$, notice $N_0 = \mathbb{Z}$, $0^{\bar{\alpha}} = 0$, $t^{\bar{0}} = 1$, and $f : N_a \rightarrow \mathbb{R}$. Here $\rho(s) = s - 1$, $\sigma(s) = s + 1$, $\nu(t) = 1$. Also define

$$\nabla_a^{-n} f(t) = \sum_{s=a}^t \frac{(t-\rho(s))^{\bar{n-1}}}{(n-1)!} f(s), \quad n \in \mathbb{N},$$

and in general

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^t \frac{(t-\rho(s))^{\bar{\nu-1}}}{\Gamma(\nu)} f(s),$$

where $\nu \in \mathbb{R} - \{-2, -1, 0\}$.

Here we set

$$\widehat{h}_\alpha(t, s) = \frac{(t-s)^{\bar{\alpha}}}{\Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

We need

Lemma 1 *Let $\alpha > -1$, $x > \alpha + 1$. Then*

$$\frac{\Gamma(x)}{\Gamma(x-\alpha)} = \frac{1}{(\alpha+1)} \left(\frac{\Gamma(x+1)}{\Gamma(x-\alpha)} - \frac{\Gamma(x)}{\Gamma(x-\alpha-1)} \right).$$

Proposition 2 Let $\alpha > -1$. It holds

$$\int_s^t \frac{(\tau-s)^{\bar{\alpha}}}{\Gamma(\alpha+1)} \nabla \tau = \frac{(t-s)^{\bar{\alpha+1}}}{\Gamma(\alpha+2)}, \quad t \geq s.$$

That is \hat{h}_α , $\alpha \geq 0$, on N_a confirm (2).

Next for $\mu, \nu > 1$, $\tau < t$, from the proof of Theorem 2.1 ([5]) we get that

$$\sum_{s=\tau}^t \frac{(t-\rho(s))^{\bar{\nu-1}}}{\Gamma(\nu)} \frac{(s-\rho(\tau))^{\bar{\mu-1}}}{\Gamma(\mu)} = \frac{(t-\rho(\tau))^{\bar{\nu+\mu-1}}}{\Gamma(\mu+\nu)},$$

where $\tau \in \{a, \dots, t\}$.

So for $t, t_0 \in N_a$ with $t_0 < t$ we obtain

$$\sum_{\tau=t_0}^t \frac{(t-\tau+1)^{\bar{\nu-1}}}{\Gamma(\nu)} \frac{(\tau-t_0+1)^{\bar{\mu-1}}}{\Gamma(\mu)} = \frac{(t-t_0+1)^{\bar{\nu+\mu-1}}}{\Gamma(\mu+\nu)}, \quad (6)$$

that is confirming (3) fractionally on the time scale $T = N_a$.

Notice also here that

$$\int_a^b f(t) \nabla t = \sum_{t=a+1}^b f(t).$$

So fractional conditions (2) and (3) are very natural and common on time scales.

For $\alpha \geq 1$ we define the time scale ∇ -Riemann-Liouville type fractional integral ($a, b \in T$)

$$J_a^\alpha f(t) = \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla \tau, \quad (7)$$

(by [8] the last integral is on $(a, t] \cap T$)

$$J_a^0 f(t) = f(t),$$

where $f \in L_1([a, b] \cap T)$ (Lebesgue ∇ -integrable functions on $[a, b] \cap T$, see [6], [7], [10]), $t \in [a, b] \cap T$.

Notice $J_a^1 f(t) = \int_a^t f(\tau) \nabla \tau$ is absolutely continuous in $t \in [a, b] \cap T$, see [8].

Lemma 3 Let $\alpha > 1$, $f \in L_1([a, b] \cap T)$. Assume that $\widehat{h}_{\alpha-1}(s, \rho(t))$ is Lebesgue ∇ -measurable on $([a, b] \cap T)^2$; $a, b \in T$. Then $J_a^\alpha f \in L_1([a, b] \cap T)$.

For $u \leq t$; $u, t \in T$, we define

$$\begin{aligned}\varepsilon(t, u) &= \int_{\rho(u)}^u \widehat{h}_{\alpha-1}(t, \rho(\tau)) \widehat{h}_{\beta-1}(\tau, \rho(u)) \nabla \tau \\ &= \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)),\end{aligned}\quad (8)$$

where $\alpha, \beta > 1$.

Next we notice for $\alpha, \beta > 1$; $a, b \in T$, $f \in L_1([a, b] \cap T)$, and $\widehat{h}_{\alpha-1}(s, \rho(t))$ is continuous on $([a, b] \cap T)^2$ for any $\alpha > 1$, that

$$J_a^\alpha J_a^\beta f(t) = \int_a^t \widehat{h}_{\alpha-1}(t, \rho(\tau)) \nabla \tau \int_a^\tau \widehat{h}_{\beta-1}(\tau, \rho(u)) f(u) \nabla u.$$

Hence

$$J_a^\alpha J_a^\beta f(t) + \int_a^t f(u) \varepsilon(t, u) \nabla u = J_a^{\alpha+\beta} f(t), \quad \forall t \in [a, b] \cap T.$$

So we have the semigroup property

$$J_a^\alpha J_a^\beta f(t) + \int_a^t f(u) \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)) \nabla u = J_a^{\alpha+\beta} f(t), \quad (9)$$

$\forall t \in [a, b] \cap T$, with $a, b \in T$.

We call the Lebesgue ∇ -integral

$$D(f, \alpha, \beta, T, t) = \int_a^t f(u) \nu(u) \widehat{h}_{\alpha-1}(t, \rho(u)) \widehat{h}_{\beta-1}(u, \rho(u)) \nabla u, \quad (10)$$

$t \in [a, b] \cap T$; $a, b \in T$, the backward graininess deviation functional of $f \in L_1([a, b] \cap T)$.

If $T = \mathbb{R}$, then $D(f, \alpha, \beta, \mathbb{R}, t) = 0$.

Putting things together we have

Theorem 4 Let $T_k = T$, $a, b \in T$, $f \in L_1([a, b] \cap T)$; $\alpha, \beta > 1$; $\widehat{h}_{\alpha-1}(s, \rho(t))$ is continuous on $([a, b] \cap T)^2$ for any $\alpha > 1$. Then

$$J_a^\alpha J_a^\beta f(t) + D(f, \alpha, \beta, T, t) = J_a^{\alpha+\beta} f(t), \quad (11)$$

$\forall t \in [a, b] \cap T$.

We make

Remark 5 Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\tilde{\nu} = m - \mu$ ($0 < \tilde{\nu} < 1$).

Let $f \in C_{ld}^m([a, b] \cap T)$. Clearly here ([10]) f^{∇^m} is a Lebesgue ∇ -integrable function.

We define the nabla fractional derivative on time scale T of order $\mu - 1$ as follows:

$$\nabla_{a*}^{\mu-1} f(t) = (J_a^{\tilde{\nu}+1} f^{\nabla^m})(t) = \int_a^t \hat{h}_{\tilde{\nu}}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla \tau, \quad (12)$$

$\forall t \in [a, b] \cap T$.

Notice here that $\nabla_{a*}^{\mu-1} f \in C([a, b] \cap T)$ by a simple argument using dominated convergence theorem in Lebesgue ∇ -sense.

If $\mu = m$, then $\tilde{\nu} = 0$ and by (12) we get

$$\nabla_{a*}^{m-1} f(t) = J_a^1 f^{\nabla^m}(t) = f^{\nabla^{m-1}}(t). \quad (13)$$

More generally, by [8], given that $f^{\nabla^{m-1}}$ is everywhere finite and absolutely continuous on $[a, b] \cap T$, then f^{∇^m} exists ∇ -a.e. and is Lebesgue ∇ -integrable on $(a, t] \cap T$, $\forall t \in [a, b] \cap T$, and one can plug it into (12).

We have

Theorem 6 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$; $f \in C_{ld}^m([a, b] \cap T)$, $a, b \in T$, $T_k = T$. Suppose $\hat{h}_{\mu-2}(s, \rho(t))$, $\hat{h}_{\tilde{\nu}}(s, \rho(t))$ to be continuous on $([a, b] \cap T)^2$.

Then

$$\int_a^t \hat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla \tau = \quad (14)$$

$$\int_a^t f^{\nabla^m}(u) \nu(u) \hat{h}_{\mu-2}(t, \rho(u)) \hat{h}_{\tilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \hat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla \tau,$$

$\forall t \in [a, b] \cap T$.

We need the nabla time scales Taylor formula

Theorem 7 ([2]) Let $f \in C_{ld}^m(T)$, $m \in \mathbb{N}$, $T_k = T$; $a, b \in T$. Then

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, a) f^{\nabla^k}(a) + \int_a^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla \tau, \quad (15)$$

$$\forall t \in [a, b] \cap T.$$

Next we present the fractional time scales nabla Taylor formula

Theorem 8 Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$; $f \in C_{ld}^m(T)$, $a, b \in T$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\tilde{\nu}}(s, \rho(t))$ to be continuous on $([a, b] \cap T)^2$. Then

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, a) f^{\nabla^k}(a) + \quad (16)$$

$$\int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\tilde{\nu}}(u, \rho(u)) \nabla u + \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla \tau,$$

$$\forall t \in [a, b] \cap T.$$

Corollary 9 All as in Theorem 8. Additionally suppose $f^{\nabla^k}(a) = 0$, $k = 0, 1, \dots, m-1$. Then

$$A(t) := f(t) - D(f^{\nabla^m}, \mu-1, \tilde{\nu}+1, T, t) \quad (17)$$

$$\begin{aligned} &= f(t) - \int_a^t f^{\nabla^m}(u) \nu(u) \widehat{h}_{\mu-2}(t, \rho(u)) \widehat{h}_{\tilde{\nu}}(u, \rho(u)) \nabla u \\ &= \int_a^t \widehat{h}_{\mu-2}(t, \rho(\tau)) \nabla_{a*}^{\mu-1} f(\tau) \nabla \tau, \end{aligned}$$

$$\forall t \in [a, b] \cap T.$$

Notice here that $D(f^{\nabla^m}, \mu-1, \tilde{\nu}+1, T, t) \in C_{ld}([a, b] \cap T)$. Also the R.H.S (17) is a continuous function in $t \in [a, b] \cap T$.

2 Fractional Nabla Inequalities on Time Scales

We present a Poincaré type related inequality.

Theorem 10 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$; $f \in C_{ld}^m(T)$, $a, b \in T$, $a \leq b$, $T_k = T$. Suppose $\widehat{h}_{\mu-2}(s, \rho(t))$, $\widehat{h}_{\tilde{\nu}}(s, \rho(t))$ to be continuous on $([a, b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 0, 1, \dots, m - 1$. Here $A(t) = f(t) - D(f^{\nabla^m}, \mu - 1, \tilde{\nu} + 1, T, t)$, $t \in [a, b] \cap T$; and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Then

$$\int_a^b |A(t)|^q \nabla t \leq \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{q}{p}} \nabla t \right) \left(\int_a^b |\nabla_{a*}^{\mu-1} f(t)|^q \nabla t \right). \quad (18)$$

Next we give a related Sobolev inequality.

Theorem 11 Here all as in Theorem 10. Let $r \geq 1$ and denote

$$\|f\|_r = \left(\int_a^b |f(t)|^r \nabla t \right)^{\frac{1}{r}}. \quad (19)$$

Then

$$\|A\|_r \leq \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{r}{p}} \nabla t \right)^{\frac{1}{r}} \|\nabla_{a*}^{\mu-1} f\|_q. \quad (20)$$

Next we give an Opial type related inequality.

Theorem 12 Here all as in Theorem 10. Additionally assume that

$$|\nabla_{a*}^{\mu-1} f| \text{ is increasing on } [a, b] \cap T. \quad (21)$$

Then

$$\begin{aligned} & \int_a^b |A(t)| |\nabla_{a*}^{\mu-1} f(t)| \nabla t \leq \\ & (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t \left| \widehat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right) \nabla t \right)^{\frac{1}{p}} \left(\int_a^b (\nabla_{a*}^{\mu-1} f(t))^{2q} \nabla t \right)^{\frac{1}{q}}. \end{aligned} \quad (22)$$

It follows related Ostrowski type inequalities.

Theorem 13 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$; $f \in C_{ld}^m(T)$, $a, b \in T$, $a \leq b$, $T_k = T$. Suppose $\hat{h}_{\mu-2}(s, \rho(t))$, $\hat{h}_{\tilde{\nu}}(s, \rho(t))$ to be continuous on $([a, b] \cap T)^2$, and $f^{\nabla^k}(a) = 0$, $k = 1, \dots, m - 1$. Denote $A(t) = f(t) - D(f^{\nabla^m}, \mu - 1, \tilde{\nu} + 1, T, t)$, $t \in [a, b] \cap T$.

Then

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq \\ & \frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \hat{h}_{\mu-2}(t, \rho(\tau)) \right| \nabla \tau \right) \nabla t \right) \|\nabla_{a^*}^{\mu-1} f\|_{\infty, [a, b] \cap T}. \end{aligned} \quad (23)$$

Theorem 14 All as in Theorem 13. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b A(t) \nabla t - f(a) \right| \leq \\ & \frac{1}{b-a} \left(\int_a^b \left(\int_a^t \left| \hat{h}_{\mu-2}(t, \rho(\tau)) \right|^p \nabla \tau \right)^{\frac{1}{p}} \nabla t \right) \|\nabla_{a^*}^{\mu-1} f\|_{q, [a, b] \cap T}. \end{aligned} \quad (24)$$

We finish general fractional nabla time scales inequalities with a related Hilbert-Pachpatte type inequality.

Theorem 15 Let $\varepsilon > 0$, $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$; $f_i \in C_{ld}^m(T_i)$, $a_i, b_i \in T_i$, $a_i \leq b_i$, $T_{ik} = T_i$ time scale, $i = 1, 2$. Suppose $\hat{h}_{\mu-2}^{(i)}(s_i, \rho_i(t_i))$, $\hat{h}_{\tilde{\nu}}^{(i)}(s_i, \rho_i(t_i))$ to be continuous on $([a_i, b_i] \cap T_i)^2$, and $f_i^{\nabla^k}(a_i) = 0$, $k = 0, 1, \dots, m-1$; $i = 1, 2$. Here $A_i(t_i) = f_i(t_i) - D_i(f_i^{\nabla^m}, \mu - 1, \tilde{\nu} + 1, T_i, t_i)$, $t_i \in [a_i, b_i] \cap T_i$; $i = 1, 2$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Call

$$F(t_1) = \int_{a_1}^{t_1} \left(\left| \hat{h}_{\mu-2}^{(1)}(t_1, \rho_1(\tau_1)) \right| \right)^p \nabla \tau_1,$$

for all $t_1 \in [a_1, b_1]$, and

$$G(t_2) = \int_{a_2}^{t_2} \left(\left| \hat{h}_{\mu-2}^{(2)}(t_2, \rho_2(\tau_2)) \right| \right)^q \nabla \tau_2,$$

for all $t_2 \in [a_2, b_2]$ (where $\widehat{h}_{\mu-2}^{(i)}$, ρ_i are the corresponding $\widehat{h}_{\mu-2}$, ρ to T_i , $i = 1, 2$).

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \nabla t_1 \nabla t_2 \leq (b_1 - a_1) (b_2 - a_2) \left(\int_{a_1}^{b_1} |\nabla_{a_1*}^{\mu-1} f_1(t_1)|^q \nabla t_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} |\nabla_{a_2*}^{\mu-1} f_2(t_2)|^p \nabla t_2 \right)^{\frac{1}{p}}. \quad (25)$$

(above double time scales Riemann nabla integration is considered in the natural interative way).

3 Applications

I) Here $T = \mathbb{R}$ case.

Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$, $f \in C^m([a, b])$, $a, b \in \mathbb{R}$.

The nabla fractional derivative on \mathbb{R} of order $\mu - 1$ is defined as follows:

$$\nabla_{a*}^{\mu-1} f(t) = (J_a^{\tilde{\nu}+1} f^{(m)})(t) = \frac{1}{\Gamma(\tilde{\nu}+1)} \int_a^t (t-\tau)^{\tilde{\nu}} f^{(m)}(\tau) d\tau, \quad (26)$$

$\forall t \in [a, b]$.

Notice that $\nabla_{a*}^{\mu-1} f \in C([a, b])$, and $A(t) = f(t)$, $\forall t \in [a, b]$.

We give a Poincaré type inequality.

Theorem 16 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $f \in C^m(\mathbb{R})$, $a, b \in \mathbb{R}$, $a \leq b$. Suppose $f^{(k)}(a) = 0$, $k = 0, 1, \dots, m - 1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_a^b |f(t)|^q dt \leq \frac{(b-a)^{(\mu-1)q}}{(\Gamma(\mu-1))^q (\mu-1) q ((\mu-2)p+1)^{q-1}} \left(\int_a^b |\nabla_{a*}^{\mu-1} f(t)|^q dt \right). \quad (27)$$

Proof. By Theorem 10. ■

We give a Sobolev type inequality.

Theorem 17 All as in Theorem 16. Let $r \geq 1$. Then

$$\|f\|_r \leq \frac{(b-a)^{\mu-2+\frac{1}{p}+\frac{1}{r}}}{\Gamma(\mu-1)((\mu-2)p+1)^{\frac{1}{p}}\left((\mu-2)r+\frac{r}{p}+1\right)^{\frac{1}{r}}}\|\nabla_{a*}^{\mu-1} f\|_q. \quad (28)$$

Proof. By Theorem 11. ■

We continue with an Opial type inequality.

Theorem 18 All as in Theorem 16. Assume $|\nabla_{a*}^{\mu-1} f|$ is increasing on $[a, b]$.

$$\int_a^b |f(t)| |\nabla_{a*}^{\mu-1} f(t)| dt \leq \frac{(b-a)^{\mu-\frac{1}{q}}}{\Gamma(\mu-1)[((\mu-2)p+1)((\mu-2)p+2)]^{\frac{1}{p}}}\left(\int_a^b (\nabla_{a*}^{\mu-1} f(t))^{2q} dt\right)^{\frac{1}{q}}. \quad (29)$$

Proof. By Theorem 12. ■

Some Ostrowski type inequalities follow.

Theorem 19 Let $\mu > 2$, $m-1 < \mu < m \in \mathbb{N}$, $f \in C^m(\mathbb{R})$, $a, b \in \mathbb{R}$, $a \leq b$. Suppose $f^{(k)}(a) = 0$, $k = 1, \dots, m-1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(a) \right| \leq \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \|\nabla_{a*}^{\mu-1} f\|_{\infty, [a, b]}. \quad (30)$$

Proof. By Theorem 13. ■

Theorem 20 Here all as in Theorem 19. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(a) \right| \leq \frac{(b-a)^{\mu-\frac{1}{q}-1}}{\Gamma(\mu-1)\left(\mu-\frac{1}{q}\right)((\mu-2)p+1)^{\frac{1}{p}}} \|\nabla_{a*}^{\mu-1} f\|_{q, [a, b]}. \quad (31)$$

Proof. By Theorem 14. ■

We finish this subsection with a Hilbert-Pachpatte inequality on \mathbb{R} .

Theorem 21 Let $\varepsilon > 0$, $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $i = 1, 2$; $f_i \in C^m(\mathbb{R})$, $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$, $f_i^{(k)}(a_i) = 0$, $k = 0, 1, \dots, m - 1$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Call

$$F(t_1) = \frac{(t_1 - a_1)^{(\mu-2)p+1}}{(\Gamma(\mu-1))^p ((\mu-2)p+1)},$$

$t_1 \in [a_1, b_1]$, and

$$G(t_2) = \frac{(t_2 - a_2)^{(\mu-2)q+1}}{(\Gamma(\mu-1))^q ((\mu-2)q+1)},$$

$t_2 \in [a_2, b_2]$.

Then

$$\begin{aligned} & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)| |f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} dt_1 dt_2 \leq \\ & (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} |\nabla_{a_1*}^{\mu-1} f_1(t_1)|^q dt_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} |\nabla_{a_2*}^{\mu-1} f_2(t_2)|^p dt_2 \right)^{\frac{1}{p}}. \end{aligned} \quad (32)$$

Proof. By Theorem 15. ■

II) Here $T = \mathbb{Z}$ case.

Let $\mu > 2$ such that $m - 1 < \mu < m \in \mathbb{N}$, $\tilde{\nu} = m - \mu$, $a, b \in \mathbb{Z}$, $a \leq b$.

Here $f : \mathbb{Z} \rightarrow \mathbb{R}$, and $f^{\nabla^m}(t) = \nabla^m f(t) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(t-k)$.

The nabla fractional derivative on \mathbb{Z} of order $\mu - 1$ is defined as follows:

$$\nabla_{a*}^{\mu-1} f(t) = (J_a^{\tilde{\nu}+1} (\nabla^m f))(t) = \frac{1}{\Gamma(\tilde{\nu}+1)} \sum_{\tau=a+1}^t (t-\tau+1)^{\tilde{\nu}} (\nabla^m f)(\tau), \quad (33)$$

$\forall t \in [a, \infty) \cap \mathbb{Z}$.

Notice here that $\nu(t) = 1$, $\forall t \in \mathbb{Z}$, and

$$\begin{aligned} A(t) &= f(t) - D(\nabla^m f, \mu-1, \tilde{\nu}+1, \mathbb{Z}, t) \\ &= f(t) - \sum_{u=a+1}^t (\nabla^m f(u)) \frac{(t-u+1)^{\mu-2}}{\Gamma(\mu-1)}, \end{aligned} \quad (34)$$

$\forall t \in [a, \infty) \cap \mathbb{Z}$.

We give a discrete fractional Poincaré type inequality.

Theorem 22 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $a \leq b$, $f : \mathbb{Z} \rightarrow \mathbb{R}$. Assume $\nabla^k f(a) = 0$, $k = 0, 1, \dots, m - 1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} & \sum_{t=a+1}^b |A(t)|^q \leq \\ & \frac{1}{(\Gamma(\mu-1))^q} \left(\sum_{t=a+1}^b \left(\sum_{\tau=a+1}^t (t-\tau+1)^{(\mu-2)p} \right) \right) \left(\sum_{t=a+1}^b |\nabla_{a*}^{\mu-1} f(t)|^q \right). \end{aligned} \quad (35)$$

Proof. By Theorem 10. ■

We continue with a discrete fractional Sobolev type inequality.

Theorem 23 Here all as in Theorem 22. Let $r \geq 1$ and denote

$$\|f\|_r = \left(\sum_{t=a+1}^b |f(t)|^r \right)^{\frac{1}{r}}.$$

Then

$$\|A\|_r \leq \frac{1}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^b \left(\sum_{\tau=a+1}^t (t-\tau+1)^{(\mu-2)p} \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \|\nabla_{a*}^{\mu-1} f\|_q. \quad (36)$$

Proof. By Theorem 11. ■

Next we give a discrete fractional Opial type inequality.

Theorem 24 Here all as in Theorem 22. Assume that $|\nabla_{a*}^{\mu-1} f|$ is increasing on $[a, b] \cap \mathbb{Z}$. Then

$$\begin{aligned} & \sum_{t=a+1}^b |A(t)| |\nabla_{a*}^{\mu-1} f(t)| \leq \\ & \frac{(b-a)^{\frac{1}{q}}}{\Gamma(\mu-1)} \left(\sum_{t=a+1}^b \left(\sum_{\tau=a+1}^t (t-\tau+1)^{(\mu-2)p} \right) \right)^{\frac{1}{p}} \left(\sum_{t=a+1}^b (\nabla_{a*}^{\mu-1} f(t))^{2q} \right)^{\frac{1}{q}}. \end{aligned} \quad (37)$$

Proof. By Theorem 12. ■

It follows related discrete fractional Ostrowski type inequalities.

Theorem 25 Let $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $a \leq b$, $f : \mathbb{Z} \rightarrow \mathbb{R}$. Assume $\nabla^k f(a) = 0$, $k = 1, \dots, m - 1$.

Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^b A(t) - f(a) \right| \leq \frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^b \left(\sum_{\tau=a+1}^t (t-\tau+1)^{\frac{\mu-2}{2}} \right) \right) \|\nabla_{a*}^{\mu-1} f\|_{\infty, [a,b] \cap \mathbb{Z}}. \quad (38)$$

Proof. By Theorem 13. ■

Theorem 26 All as in Theorem 25. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{1}{b-a} \sum_{t=a+1}^b A(t) - f(a) \right| \leq \frac{1}{(b-a)\Gamma(\mu-1)} \left(\sum_{t=a+1}^b \left(\sum_{\tau=a+1}^t (t-\tau+1)^{\frac{(\mu-2)p}{2}} \right)^{\frac{1}{p}} \right) \|\nabla_{a*}^{\mu-1} f\|_{q, [a,b] \cap \mathbb{Z}}. \quad (39)$$

Proof. By Theorem 14. ■

We finish article with a discrete fractional Hilbert-Pachpatte type inequality.

Theorem 27 Let $\varepsilon > 0$, $\mu > 2$, $m - 1 < \mu < m \in \mathbb{N}$; $i = 1, 2$; $f_i : \mathbb{Z} \rightarrow \mathbb{R}$, $a_i, b_i \in \mathbb{Z}$, $a_i \leq b_i$. Suppose $\nabla^k f_i(a_i) = 0$, $k = 0, 1, \dots, m - 1$. Here $A_i(t_i) = f_i(t_i) - \sum_{u_i=a_i+1}^{t_i} (\nabla^m f(u_i))^{\frac{(t_i-u_i+1)^{\frac{\mu-2}{2}}}{\Gamma(\mu-1)}}$, $\forall t_i \in [a_i, \infty) \cap \mathbb{Z}$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Call

$$F(t_1) = \sum_{\tau_1=a_1+1}^{t_1} \frac{(t_1-\tau_1+1)^{\frac{(\mu-2)p}{2}}}{(\Gamma(\mu-1))^p},$$

$\forall t_1 \in [a_1, \infty) \cap \mathbb{Z}$, and

$$G(t_2) = \sum_{\tau_2=a_2+1}^{t_2} \frac{(t_2 - \tau_2 + 1)^{(\mu-2)q}}{(\Gamma(\mu-1))^q},$$

$\forall t_2 \in [a_2, \infty) \cap \mathbb{Z}$.

Then

$$\begin{aligned} & \sum_{t_1=a_1+1}^{b_1} \sum_{t_2=a_2+1}^{b_2} \frac{|A_1(t_1)| |A_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \leq \\ & (b_1 - a_1)(b_2 - a_2) \left(\sum_{t_1=a_1+1}^{b_1} |\nabla_{a_1*}^{\mu-1} f_1(t_1)|^q \right)^{\frac{1}{q}} \left(\sum_{t_2=a_2+1}^{b_2} |\nabla_{a_2*}^{\mu-1} f_2(t_2)|^p \right)^{\frac{1}{p}}. \end{aligned} \quad (40)$$

Proof. By Theorem 15. ■

We intend to publish the complete article with full proofs elsewhere.

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